SUFFICIENT CONDITIONS FOR THE ABSOLUTE EXTREMUM IN A VARIATIONAL PROBLEM OF THE BOLZA – MEYER TYPE

(DOSTATOCHNYE USLOVIIA ABSOLIUTNOGO EKSTREMUMA V ODNOI VARIATSIONNOI ZADACHE TIPA BOL'TSA – MAIERA)

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A solution of a variational problem of the Bolza-Meyer type based on the requirement of sufficient conditions for the absolute minimum [1] is investigated.

Self-similarity of the class of problems under consideration is based on the fact that some functions entering the equations are parametric and dependent in a well known manner on the time and position of the points of discontinuity of the first kind.

Papers [2 and 3] show that the above problem is a variational one, and using the classical definition of variation they derive the necessary conditions for the extremum of the resulting functional.

Application of the optimality principle helps us to establish the existence of some non-trivial curves, on which the absolute extremum of a given functional is reached. Thus, for example, the absolute extremum of the functional considered on the class of control functions representing the function of variation of the relative mass of a multi-stage rocket, can perhaps be only achieved on the representation of a rocket with the infinite number of stages (continuous).

The formulas obtained (5.2) represent the Tsiolkovski formula (5.4) generalised for the case of arbitrary motion of a rocket under continuous thrust. A well known problem allied to the present one is that belonging to the dynamics of flight, and it deals with the programming of the thrust of the reaction engine when it is postulated that the engine performs under the conditions of intermittent maximum thrust.

Absolute extremum of the functional is reached, in this case, on the curves realised under the gliding conditions.

1. Let the mass of a multi-stage rocket decrease linearly [4], and let $y = m/m_0$ be the dimensionless mass of the multi-stage rocket, where m_0 is the mass of the composite rocket at the moment of start. If we consider the auxiliary plane y, t, then the parts of the curve $y = y(t)$ corresponding to the intervals of the powered flight will be represented by the sloping straight lines, while the intervals during which the empty stage separates,

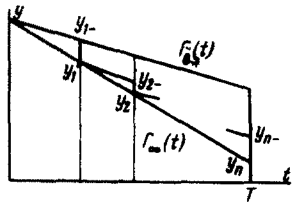

FIG. 1

will be represented by the vertical segments (Fig. 1).

We shall assume that the performance of the engines of consecutive stages is not discontinuous, and we shall denote the ration of the 'dead weight' of the i-th stage to the mass of the fuel contained by it, by k_i. Also, we shall have

$$y(t_i - 0) = y_{i-}, \qquad y(t_i + 0) = y_i \qquad (i = 1, \ldots, n)$$

where t_i is the moment of separation of the i-th stage. Analytically, the function y is defined by $2n$ equations of the type

$$y_i = y_{i-1} - \beta_i (t_i - t_{i-1}), \quad y_i = y_{i-}(1 + k_i) - k_i y_{i-1} \qquad (1.1)$$

where

$$t_0 = 0, \quad t_n = T, \quad y_0 = 1, \quad y_n = m_p / m_0$$

Here T is the time of working of the engines, m_p is the payload and β_i is the consumption of fuel per second by the i-th stage.

From (1.1) it follows that the dimensionless mass of the composite rocket plays the part of a control function of a particular type as far as the form of the control function between its points of discontinuity of the first kind is known. Points t_i 'floating' on the segment $[t_0, T]$ influence the magnitude of the resulting functional.

Let us consider the motion of a single-stage device under the assumption that the time of flight of this device is greater, then the time of its powered flight.

For simplicity we shall assume that only one pause is allowed during the active period of the engine. The relative mass changes in a linear manner and the ratio of the thrust P to the initial weight of the device is equal to

$$P / m_0 = -V_r y^{\cdot}.$$

The control function $y(t)$ and its derivative are given, in our case, by the following analytical equalities (Fig. 2)

$$\begin{aligned} y_1 &= 1 - \beta_1 (t - t_0), & y_1^{\cdot} &= -\beta_1 & (0 \leqslant t \leqslant t_1) \\ y_2 &= 1 - \beta_1 (t_1 - t_0), & y_2^{\cdot} &= 0 & (t_1 < t \leqslant t_2) \\ y_3 &= y_2 - \beta_2 (t - t_2), & y_3^{\cdot} &= -\beta_2 & (t_2 < t \leqslant T) \end{aligned} \qquad (1.2)$$

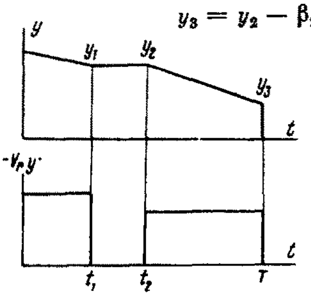

FIG. 2

This control function $y(t)$ belongs, together with its derivative, to the class of functions the behavior of which between the points of discontinuity of the derivative, is known. The discontinuities of the derivative are determined from the conditions of the extremum of an arbitrary functional.

2. Let the process occurring in some dynamical system be described by n ordinary differential equations of the first order

$$x_s^{\cdot} - f_s(x, u, y, t) = 0 \qquad (s = 1, \ldots, n) \quad (2.1)$$

where $x(t)$ and $u(t)$ are the n- and r-dimensional vector functions respectively, the properties of which are described in [1], and $y(t, t_i)$ is the l-dimensional vector function, the components $\{y_1(t,t_i), \ldots, y_l(t,t_i)\}$, of which are continuous almost everywhere on the interval $[t_0, T]$ with the exception of the points t_i, on which these functions possess discontinuities of the first kind. For any fixed $t \in [t_0, T]$, the vector function $y(t, t_i)$ belongs to a finite closed manifold Ω.

Conditions imposed on $x(t)$, $u(t)$ and $y(t, t_i)$ define a set $V(t, t_i)$ of admissible values of the aggregates of $n + r + i$ numbers (x_s, u_j, t_i) for every $t \in [t_0, T]$, and a region B of admissible values in the (n + 1-dimensional) space t, x together with the set V of admissible aggregates of $(n + r + i + 1)$ numbers x_s, u_j, t, t_i $(s = 1, \ldots, n;\ j = 1, \ldots, r)$.

We shall now pose a problem on determining the minimum of the functional

$$J = g(x_0, x_1) + \int_{t_0}^{T} f^0(t, x, u, y)dt \tag{2.2}$$

on the space of triads of vector functions $x(t)$, $u(t)$, $y(t, t_i)$. The set of the triads of vector functions possessing the properties given above and satisfying (2.1), belongs to the class D_i; here and in the following, the index i means that the variational problem is considered on the class of functions $y(t, t_i)$ which possess, on the interval $[t_0, T]$ not more than i discontinuities of the first kind.

Obviously, if we determine the points t_i, we shall know the vector function $y(t, t_i)$.

3. The fundamental theorem of [1] has the following form.

Theorem 1. Let a sequence $\{x_s^i, u_s^i, y_s^i\}$ of vector functions be given. The sufficient condition for this sequence to minimise the functional J on D_i is, that there exists a function $\varphi(t, x)$, such, that

$$\begin{gathered} R[t, x_s^i, u_s^i, y_s^i] \to \sup R = \mu(t, t_i), \quad [x, u, y] \in V(t, t_i),\ t \in [t_0, T] \\ \Phi[x_{0s}^i, x_{1s}^i] \to \inf \Phi(x_0, x_1), \quad x_0 \in B(t_0), \quad x_1 \in B(T) \\ R(t_i - 0) - R(t_i + 0) + \int_{t_0}^{T} \partial R / \partial t_i dt = 0 \end{gathered} \tag{3.1}$$

If the minimum on D_i exists, then the first and second condition of (3.1) have the form

$$R[t, x^*, u^*, y] = \mu(t, t_i), \quad \Phi(x_0, x_1) = \inf \Phi(x_0, x_1), \quad x_0 \in B(t_0), \quad x_1 \in B(T)$$

These conditions coincide with the conditions of the fundamental theorem (1). We shall now prove the validity of the third condition. Let us construct the functional

$$I = \Phi(x_0, x_1) - \int_{t_0}^{T} R(t, x, u, y)\, dt \tag{3.2}$$

This functional coincides on the set D_i with J by virtue of the properties of the function R. If $y \in \Omega$, then the functional I is a function of the points t_i. Writing (3.2) as

$$I = \Phi(x_0, x_1) - \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} R_i(t, x, u, y)\, dt \tag{3.3}$$

and equating the derivative of I with respect to t_i to zero, we obtain the third condition. We assume that the curve minimising the functional exists. Various methods of determination of the extremum depending on the method of selecting the function $\varphi(t, x)$, are possible.

We shall consider the method of solution of the stated problem, which uses Lagrange's formal approach.

Let the region $B(t)$ of permissible phase coordinates be open, and $x(t_0)$ be the given point of the phase space when $t = t_0$. Then (3.1) will have the form

$$\begin{gathered} R_{x_s}(t, x^*, u^*, y) = 0 \qquad (s = 1, \ldots, n) \\ \varphi_x f(t, x^*, u^*, y) - f^0(t, x^*, u^*, y) = \sup, \qquad u \in Q(t) \\ H(t_i - 0) - H(t_i + 0) + \int_{t_i}^{T} \partial H / \partial t_i dt = 0 \end{gathered} \tag{3.4}$$

Introducing the vector function $\lambda(t) = \varphi_x[t, x^*]$ the components of which are $\lambda_s(t)$ $(s = 1, \ldots, n)$, we shall represent the conditions in the form

$$\lambda_s^{\cdot} + \partial H / \partial x_s = 0, \quad H(t, x^*, u^*, y) = \sup R(t, x, u, y),\ u \in Q(t) \tag{3.5}$$

If the dynamic system is such that its behavior depends only on $y(t, t_i)$, then the equations (3.4) will define the required instants of time [3 and 4].

4. We shall now determine the number of points of discontinuity necessary for the absolute extremum of the functional J to occur.

Restating the problem more accurately, we shall let the functional (2.2) be, on some set M, bounded from below

$$\inf J = m > -\infty$$

Let the function $R(t, x, u, y)$ possess, for all $t \in [t_0, T]$ and any y, a unique supremum in the space $x(t)$, $u(t)$, and let it be bounded for any x, u and y.

The manifold Ω will be bounded and closed. If the absolute extremum of the functional J is not realised on the boundary of the manifold Ω, then it will be realised on its closure.

Indeed, let us consider a system of expanding manifolds $\{D_i\}$ $(i = 1, \ldots, \infty)$, corresponding to i points of discontinuity of the first kind within the values $y(t, t_i)$. Let us define, on each manifold D_i, the magnitude

$$\mu(t, t_i) = \sup R(t, x, u, y), \ [x, u, y] \in V \tag{4.1}$$

The sequence $\{\mu_i\}$ cannot increase, since the dimensionality of the manifold D_i increases on the transition from the control functions with the smaller number of discontinuities of the functions possessing a larger number of them.

Since according to the previous assumption the sequence under discussion is infinite, non-decreasing and bounded, hence it has the limit

$$\lim_{i\to\infty} \mu(t, t_i) = \mu(t), \quad \mu(t) = \sup R(t, x, u, y), \quad [x, u, y] \in V \tag{4.2}$$

We shall consider two possible modes of behavior of the sequence $y(t, t_i)$ when $i \to \infty$. Let the components of the vector function $y(t, t_i)$ form, when $i \to \infty$, a Cauchy sequence. In this case there exists a unique limit function $y^* = \lim y(t, t_i)$ when $i \to \infty$. Function y^*

satisfies the initial system of equations. The absolute extremum of the function R is reached on the triad of vector functions $x^*(t)$, $y^*(t)$ and $u^*(t)$. We shall now assume that, when the number of points t_i increases without bounds, the values of the vector function $y(t, t_i)$ tend to two limiting values equal to y_1 and y_2 respectively. The sequence of supremums (4.1) of the function R being non-decreasing and bounded, tends to the unique limit, i.e.

$$\sup R(t, x, u, y_1) = \sup R(t, x, u, y_2) \to \mu(t) \tag{4.3}$$

We shall call the curve x^*, u^*, $y_1 = y_2$ in the space t, x the curve of zero distance of the gliding mode. It does not satisfy the initial system of equations [5], but nevertheless continuous functions $\alpha_1(t)$ and $\alpha_2(t)$ can be found such, that the given curve will satisfy the relations

$$x_s^{*\cdot} = \alpha_1 f(x^*, u, y_1, t) + \alpha_2 f(x^*, u, y_2, t)$$
$$1 = \alpha_1 + \alpha_2, \quad \alpha_1 \geqslant 0, \quad \alpha_2 \geqslant 0, \quad \sup R(t, x, u, y), \quad u \in Q(t, x) \tag{4.4}$$

Various methods of constructing the minimising sequence are possible, and they depend on the choice of $\varphi(t, x)$.

Using Lagrangian notation, let us write the equations giving the solution of the stated problem assuming, that the function R has, for all $t \in [t_0, T]$ and any y, unique supremum points in the space $x(t)$, $u(t)$. The sequence y_i has, for $i \to \infty$ the limit values y_k $(k = 1, 2)$, and the initial system of equations assumes the following form

$$\lambda_s^{\cdot} + \alpha_1 \partial H / \partial x_s (x, u, y_1, t) + \alpha_2 \partial H (x, u, y_2, t) / \partial x_s = 0 \qquad (s = 1, \ldots, n)$$
$$H(t, \lambda, u, y_k) = \sup, \qquad u \in Q(t) \qquad (k = 1, 2) \tag{4.5}$$
$$x^{*\cdot} = \alpha_1 f(x, u, y_1, t) + \alpha_2 f(x, u, y_2, t), \qquad 1 = \alpha_1 + \alpha_2, \quad \alpha_1 \geqslant 0, \quad \alpha_2 \geqslant 0$$

5. We shall consider the motion of a multi-stage rocket. Equation of motion of the Center of Gravity of such a device can be represented by (2.1). We assume the control function of the dimensionless mass $y = m/m_0$ to be known (see par. 1). We can easily come to the conclusion that the given function $y(t, t_i)$ lies, for any i, within the closed region, the boundaries of which satisfy the equations

$$\Gamma_0 = 1 - \beta t, \qquad \Gamma_\infty(t) = 1 - \beta(k + 1)t \tag{5.1}$$

The upper bound of this set corresponds to the flight of a single-stage rocket, while the lower to the rocket in powered flight. Consequently, the extremum of the functional can be realised on the boundaries definable by (5.1).

Equations of motion of the rocket in powered flight become in this case,

$$x_s^{\cdot} - f_s(x, u, \Gamma_\infty(t), t) = 0 \tag{5.2}$$

or, if $u(t)$ is chosen according to a predetermined program i.e. $u = u(t)$, then the system of differential equations (5.2) will assume the form

$$x_s^{\cdot} - f_s(x, \Gamma_\infty(t), t) = 0$$

Let us consider the Tsiolkovski problem on the vertical ascent of a composite rocket in a homogenous gravity field. Equation of motion of the Center of Gravity of this rocket will be

$$v^{\cdot} = -V_r^{\cdot} y^{\cdot} / y - g \tag{5.3}$$

where v is the velocity of the Center of Gravity of the composite rocket and $V_r = \text{const}$ is the relative exhaust velocity.

The limiting velocity V^* achieved under the conditions of continuous thrust is

$$\frac{V^*}{V_r} = \int_{\Gamma_\infty (t)} \frac{\beta}{v}\, dt - \frac{gT}{V_r} = \frac{1}{1+k} \ln \frac{m_0}{m_p} - \frac{gT}{v_r} \tag{5.4}$$

Hence, we have determined V^* without obtaining a proper solution of (5.3). For comparison, we shall show solution of a similar problem using the previous method. Having found the solution of (5.3) we shall have

$$y_n = \{(1 + k) \exp [- (v / nV_r + gT / nV_r)] - k\}^n$$

which, on passing to the limit when $n \to \infty$, gives

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} [(1 + k)(1 - I / n) - k]^n = \lim_{n\to\infty} \{[1 - (1 + k)\, y / n]^{-n / I(1+k)}\}^{-I(1+k)} =$$
$$= \exp [I (1 + k)], \qquad I = (v + gT) / V_r$$

Hence

$$V^* = [V_r / 1 + k] \ln m_0 / m_p - gT$$

In conclusion, we shall consider the vertical ascent of a rocket under the continuous thrust in the homogeneous medium and in the homogeneous gravity field.

Equation of motion will, in this case, be

$$v^{\cdot} = - g - c_x \rho S v^2 / 2m_0 [1 - \beta (1+k)t] + \beta V_r / [1 - \beta (1 + k)t], \quad D = {}^1/_2 c_x \rho S v^2 \tag{5.5}$$

Here D is the drag, c_x is the coefficient of drag, S is the cross-section area of the middle part of the rocket and ρ is the density of the medium in which the flight takes place. The solution of (5.4) can be represented in the form

$$v = \frac{1}{2} (a_0 y / a_1)^{1/2} \left\{\frac{c\,[J_{\nu+1}(\xi) - J_{\nu-1}(\xi)] + Y_{\nu+1}(\xi) - Y_{\nu-1}(\xi)}{cJ_\nu(\xi) + Y_\nu(\xi)}\right\}$$

$$\nu = 2 (a_1 a_0)^{1/2}, \qquad \xi = 2 (a_1 a_0 y)^{1/2}, \qquad a_0 = g / \beta (1 + k), \qquad a_1 = c_x \rho S / 2m_0 \beta (1 + k)$$

Assuming that $\xi_0 = 2 \sqrt{a_1 a_0}$, $v_0 = 0$ when $t_0 = 0$, we shall find

$$c = [Y_{\nu-1}(\xi_0) - Y_{\nu+1}(\xi_0)] / [J_{\nu+1}(\xi_0) - J_{\nu-1}(\xi_0)]$$

Let us now consider the motion of a hypothetical rocket possessing the following characteristics: $k = 0.1$, $y_n = 0.1$, $\beta = 0.005\ \text{sec}^{-1}$, $V_r = 3000$ m/sec., $m_0 = 165.3$ t., $T = 164$ sec., $c_x = 1/3$, $S = 5\ \text{m}^2$ and $\rho = 0.01\ \text{kg/sec}^2/\text{m}^4$.

In this case the upper limit $V^* = 4167$ m/sec.

If the model of a single-stage rocket is used, then the velocity at the moment of burn-out $V = 3134$ m/sec.

Our next case is that of a horizontal flight of a winged, single-stage, reaction device, the engine of which may, in this case, work intermittently. Then, the extremum of the functional is achieved on the curves possessing an infinite number of points of transition from one mode of flight, to the other. Using the equations (4.5), let us derive the equation of the gliding curve. Equations of motion of the plane are

$$v^{\cdot} = - [D (v, y) - V_r y_2] / y, \qquad y^{\cdot} = y_2 \tag{5.6}$$

where y_2 assumes the values $(0, -\beta)$. The function H has the form

$$H = -\lambda_1 [D(v, y) - V_r y_2] + \lambda_2 y_2$$

The condition $H(y_2 = 0) = H(y_2 = -\beta)$ results in

$$\lambda_1 V_r / y + \lambda_2 = 0 \tag{5.7}$$

Along the gliding curve, the following equations are valid

$$v^{\cdot} = -[D(v, y) - y^{\cdot} V_r] / y$$

$$\lambda_2^{\cdot} - \lambda_1 [D - yD_y + y^{\cdot} V_r] / y^2 = 0, \quad \lambda_1^{\cdot} - \lambda_1 D_v / y = 0$$

Differentiating (5.7) with respect to time and taking into account (5.6), we shall obtain

$$V_r D_v + D - yD_y = 0 \tag{5.8}$$

which is a well known equation defining the motion of a plane under a smooth thrust. If we assume that the thrust can be regulated, then we shall find, that the curve (5.8) is a part of the absolute minimal curve [1].

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